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RADIATIVE TRANSPORT IN ELECTRIC ARCS

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RADIATIVE TRANSPORT IN ELECTRIC ARCS

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13. ABSTRACT

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FORWORD

This is the final report under Contract F33615-71-C-1289 for the Aerospace Research Laboratories, Air Force System Command, United States Air Force. It covers the period from 1 February 1971 to 30 November 1972. Two journal articles have also been written under the sponsorship of this contract "Electrical Conductivity from Electric Arc Measurements", by R. S. Devoto and D. Mukherjee, to be published in Journal of Plasma Physics and "Measurement of Argon Electrical Conductivity at High Pressure", by U. Bauder, R. S. Devoto and D. Mukherjee, submitted to Physics of Fluids. The senior author of this report is now located at the School of Mechanical Engineering, Georgia Institute of Technology. Technical Monitors for this report were Dr. U. Bauder, Thermodynamics Laboratory, and Capt. Allen M. Hunter, II, Plasma Physics Laboratory, Aerospace Research Laboratory.

ABSTRACT

Equations are developed which govern the behavior of a wall-stabilized electric arc in which heat transfer takes place via conduction and radiation emission and reabsorption within the heated gas. The differential approximation is used to describe the radiative transport. Several numerical schemes are described which were used in an attempt to integrate these equations; none were found to work for all values of the absorption coefficient. A discussion of the development of higher-order differential approximations is also included.

TABLE OF CONTENTS

SECTION I

INTRODUCTION

An important need for application to hypersonic wind tunnels is a source of high-pressure (to 1000 atm) high temperature (to 15000°K) air. The high pressure electric arc would satisfy this need; its properties are as yet only imperfectly known. For this reason a program was undertaken to compute properties such as temperature profiles, power input, etc. of a particular shape of high-pressure arc, namely a wall-stabilized cylindrically symmetric arc.

At high pressures, account must be taken of energy transport via radiation reabsorption within the arc as well as by the traditional thermal conduction. However, the exact theory of radiative transport is complicated, and therefore difficult to apply in real problems. For this reason many people have employed the so-called differential approximation. The computation of arc properties with this approximation is described in Section II. Because of doubts about the accuracy of this approximation for certain values of the optical depth, we have also investigated higher-order methods. This work is described in Section III.

SECTION II

LOWEST-ORDER DIFFERENTIAL APPROXIMATION

For a cylindrically symmetric arc, the gas energy equation and the D1 equations for radiative transfer can be written in the form

$$\frac{1}{r}\frac{d}{dr}(rq^T) = \sigma E^2 \tag{1}$$

$$\frac{dT}{dr} = -\frac{1}{\lambda}(q^T - q^R) \tag{2}$$

$$\frac{1}{r}\frac{d}{dr}(rq_\nu^R) = c\varkappa_\nu(u_\nu^* - u_\nu) \tag{3}$$

$$\frac{du_\nu}{dr} = -\frac{3\varkappa_\nu}{c}q_\nu^R \tag{4}$$

where q^T and q^R are the total and radiative contributions to the heat flux, $\varkappa_\nu$ is the effective absorption coefficient, u_ν is the radiation energy density, and u_ν^* is its equilibrium value, given by

$$u_\nu^* = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1} \tag{5}$$

Other symbols have their usual meaning.

Boundary conditions for these equations can be written in several equivalent ways. In the present case we take

$$T(0) = T_A \tag{6}$$

$$q^T(0) = 0 \tag{7}$$

$$q^R(0) = 0 \tag{8}$$

$$T(R) = T_W \tag{9}$$

$$cu_\nu(R) = 2q_\nu^R(R) \tag{10}$$

Equations (6) - (9) are obvious on physical grounds; Eq. (10) arises from the necessity for continuity of heat flux at the arc outer boundary.

We see that these are four coupled first order differential equations, but with five boundary conditions, seemingly one too many. One of the boundary conditions is used in determining the value of E, so this is an eigenvalue problem. In an alternative formulation, E is given and $T(0)$ is computed as part of the solution, so Eq. (6) is no longer needed.

The above equations can be transformed into a form more convenient for solution if we define new dimensionless variables

$$x \equiv (r/R)^2 \tag{11}$$

$$u \equiv T/T_A \tag{12}$$

$$v \equiv \frac{R}{2T_A\lambda_A} x^{\frac{1}{2}} q^T \tag{13}$$

$$w_\nu \equiv \frac{u_\nu}{u^*_{\nu A}} \tag{14}$$

$$z_\nu \equiv \frac{3\tau_{\nu A}}{2cu^*_{\nu A}} x^{\frac{1}{2}} q^R_\nu \tag{15}$$

$$\tau_{\nu A} \equiv R\varkappa_{\nu A} \tag{16}$$

$$\alpha = \frac{R^2E^2\sigma_A}{4T_A\lambda_A} \tag{17}$$

Equations (1) - (4), (6) - (10) then become

$$\frac{du}{dx} = -\frac{\lambda_A}{x\lambda}\left[v - \frac{c}{3\lambda_A T_A}\int_0^\infty \frac{u_{\nu A}^* z_\nu d\nu}{\varkappa_{\nu A}}\right] \tag{18}$$

$$\frac{dv}{dx} = \alpha\frac{\sigma}{\sigma_A} \tag{19}$$

$$\frac{dw_\nu}{dx} = - \frac{\varkappa_\nu}{x\varkappa_{\nu A}} z_\nu \tag{20}$$

$$\frac{dz_\nu}{dx} = \frac{3\tau_{\nu A}\varkappa_\nu}{4\chi_{\nu A}} \tag{21}$$

$$u(0) = 1 \tag{22}$$

$$v(0) = 0 \tag{23}$$

$$z_\nu(0) = 0 \tag{24}$$

$$u(1) = T_W/T_A \tag{25}$$

$$w_\nu(1) = \frac{4z_\nu(1)}{3\tau_{\nu A}} \tag{26}$$

at $x = 0$, the limiting forms of Eqs. (18) and (20) are

$$\frac{du}{dx} = - \frac{\lambda_A}{\lambda} \left[\frac{dv}{dx} - \frac{c}{3\lambda_A T_A} \int_0^\infty \frac{u_{\nu A}}{\varkappa_{\nu A}} \frac{dz_\nu}{dx} d\nu \right] \tag{18a}$$

$$\frac{dw_\nu}{dx} = - \frac{\varkappa_\nu}{\varkappa_{\nu A}} \frac{dz_\nu}{dx} \tag{20a}$$

which must be used near $x = 0$ to avoid the singularities in Eqs. (18) and (20).

Preparatory to solving the full set of equations above we can look at several limiting cases.

1. Negligible Radiation - In this case the equations become the well-known Elenbaas-Heller equation. In the form of the equations used here, the mathematical problem is a two-point boundary value problem i.e., the two first-order (or one second-order) equations must be solved subject to conditions at two boundaries. In the present case we can numerically integrate the differential equations (18)-(19) as an initial-value problem using the boundary conditions (22)-(23) as initial values. A value is assumed for

α (say α_1) and the equations integrated to $x = 1$. If Eq. (25) is satisfied to within a small constant ϵ, then α_1 is the eigenvalue and the computed solution is the desired one. In general, Eq. (25) will not be satisfied and we must guess a new value for α, say α_2. The procedure for computing the new value of α must be chosen with care since a great deal of computation time can be used while integrating the differential equation from $x = 0$ to $x = 1$ with new values of α. In the present case, after the initial guess α_1, α_2 was computed from

$$\alpha_2 = \alpha_1 + \begin{cases} 0.01\ \alpha_1\ , & u(1;\alpha_1) < T_W/T_A \\ -0.01\ \alpha_1, & u(1;\alpha_1) > T_W/T_A \end{cases} \tag{27}$$

and further α_ν from

$$\alpha_\nu = \alpha_{\nu-1} - \frac{\alpha_{\nu-1} - \alpha_{\nu-2}}{u(1;\alpha_{\nu-1}) - u(1;\alpha_{\nu-2})} \left| u(1,\alpha_{\nu-1}) - T_W/T_A \right| \tag{28}$$

The dependence of u on α_ν is explicitly indicated above. An example of a temperature profile computed for argon at 1 atm pressure with $T_A = 10^4\,^oK$ and $T_W = 10^3\,^oK$ is given in Fig. 1. Shown is

$$\theta = \frac{T - T_W}{T_A - T_W}$$

vs. x rather than u(x).

2. <u>Constant Effective Absorption Coefficient</u> - In this case Eqs. (18) - (22) reduce to the following

$$\frac{du}{dx} = -\frac{\lambda_A}{x\lambda}\left(v - \frac{1}{N} z\right) \tag{29}$$

$$\frac{dv}{dx} = \alpha \frac{\sigma}{\sigma_A} \tag{30}$$

$$\frac{dw}{dx} = - \frac{z}{x} \tag{31}$$

$$\frac{dz}{dx} = \frac{3\tau^2}{4} (w^* - w) \tag{32}$$

with correspondingly simplified boundary conditions. N is the so-called conduction-radiation parameter and is given by

$$N = \frac{\lambda_A \ \varkappa}{4\bar{\sigma} T_A^3} \tag{33}$$

where $\bar{\sigma}$ is the Stefan-Boltzmann constant, and is, crudely speaking, a measure of the relative importance of conduction and radiation heat transfer in the arc. Other quantities are $\tau \equiv R\varkappa$ (a constant) and $w^* = u^4$.

As a preliminary to integrating the arc equation with varying $\varkappa_\nu$, the solution of these equations was programmed for the IBM 360. Important to the program are two major parts: (1) the method for numerically integrating the differential equations, (2) the method for satisfying simultaneously the boundary conditions at $x = 0$ and $x = 1$.

In view of the fact that the differential equations probably become stiff for $\tau \gtrsim 1$, it is necessary to use a technique applicable to such equations. A program for integrating systems of stiff or non-stiff equations has been recently published by Gear[1] and was used for the present case. It was found, however, that under certain conditions the Gear program would not integrate the equations and exited with an error message. Several months of effort failed to correct this problem and it was only recently learned that this program has been rewritten by Gear to remove this error. The new Gear program has since been incorporated into the program and has given no more difficulties.

The second problem listed above has proved much more difficult and was not resolved in this work. For Eqs. (29)-(32) there are boundary conditions

to be satisfied by u(as in the Elenbaas-Heller equation) and by z and w. To integrate the equations as an initial-value problem we must guess α and $w(0)$, integrate to $x = 1$ and see if Eqs. (25) and (26) are satisfied. If they are not satisfied, then α and $w(0)$ must somehow be adjusted in an iterative process to do so. As a first attempt to iterate, after initial guesses, α and $w(0)$ were adjusted as in Eqs. (27)-(28). This procedure was successful for small τ, but did not converge for $\tau > 0.01$. Profiles of θ vs x for $\tau = 0.0025$, 0.01 are shown in Fig. 1 for an argon arc at $T_A = 10^4$ °K, $p = 1$ atm. We note that the temperature profile is depressed near the axis due to radiative losses, as we would expect.

Since the above procedure was not successful for $\tau > 0.01$, a new method was sought. It turns out that the radiation density at the arc center can be evaluated from an exact, fairly simple two-fold quadrature once the temperature profile (or an approximation thereto) is known. In the notation of this report the general expression reads[2,3]

$$w_\nu(0) = R\int_0^1 \varkappa_\nu(x)\, w_\nu^*(x)\, D_1\left[\tau_\nu(0,x)\right] dx \tag{34}$$

where

$$\tau_\nu(0,x) = \int_0^x \varkappa_\nu(x')dx' \tag{35}$$

is the optical depth at frequency ν and position x and $D_1(x)$ is an integrated Bessel function,

$$D_1(x) \equiv \int_0^{\pi/2} e^{-\frac{x}{\cos\theta}}\, d\theta = \int_x^\infty K_0(y)dy \tag{36}$$

For constant $\varkappa$ Eq. (34) reduces to

$$w(0) = \tau\int_0^1 u^4(x)\, D_1(\tau x)dx \tag{37}$$

with $\tau \equiv R\varkappa$.

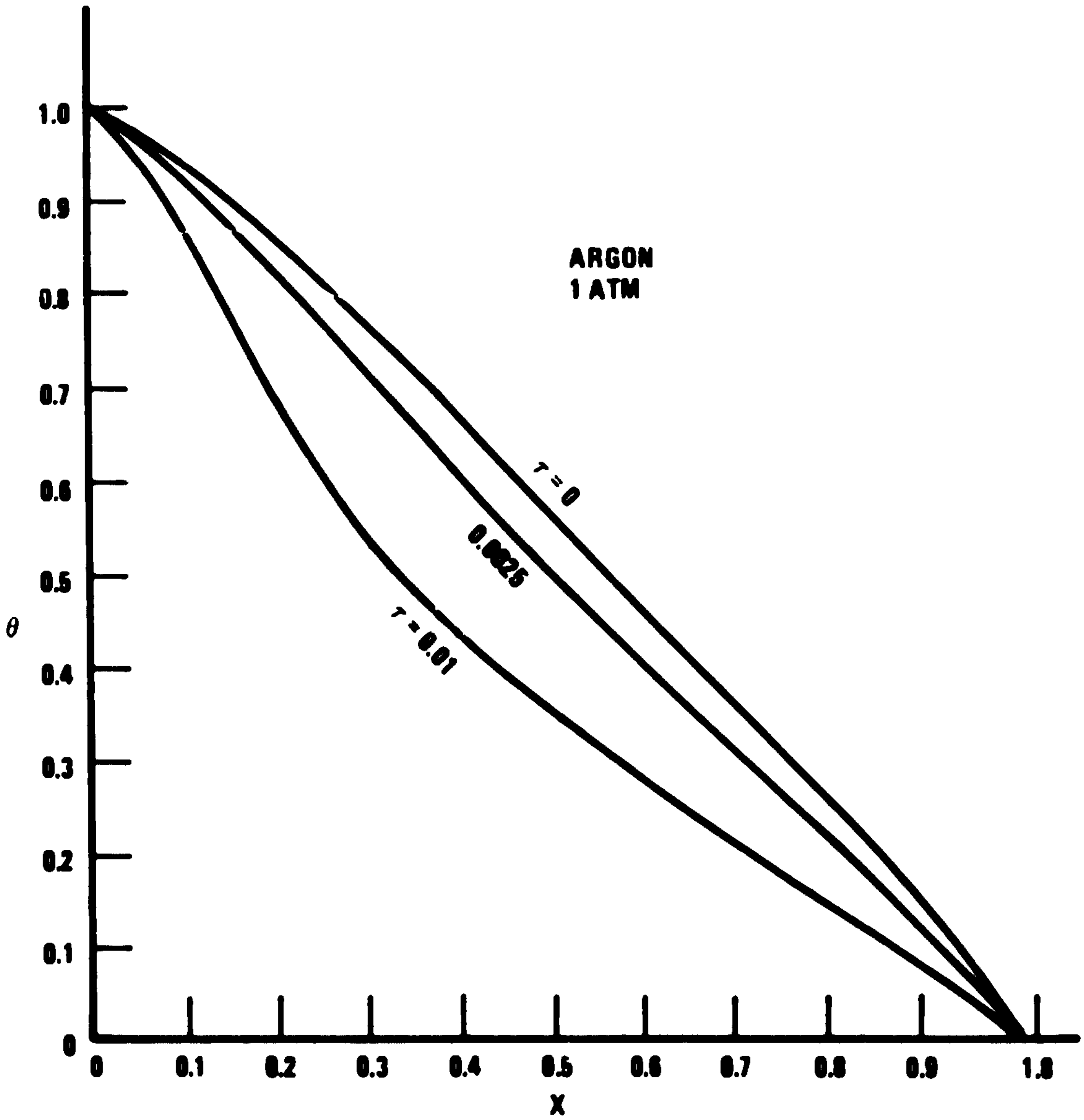

Figure 1. Dimensionless Temperature Profile of Argon Arc at 1 atm ($T_A = 10^4$, $T_W = 10^3$°K).

The new interation procedure is as follows:

(1) Compute $u(x)$ without radiation <u>or</u> feed in assumed $u(x)$;

(2) Compute $w(0)$ from Eq. (37);

(3) With $w(0)$ from step (2) and guessed α_1 integrate arc equations until convergence is obtained on α, i.e., the eigenvalue α for the $w(0)$ computed in (2) is obtained;

(4) Recompute $w(0)$ with Eq. (37) and repeat (3);

(5) After $w(0)$ no longer changes, check to see if Eq. (26) is satisfied. If so, this is the desired solution; if not, then apply a scheme similar to Eqs. (27)-(28) to both α and $w(0)$ until convergence is obtained. Steps (1)-(4) should ensure that the initial α and $w(0)$ are close enough to the true solution to obtain convergence.

This procedure was tried for $\tau > 0.01$ but it also did not converge. The reason for the failure probably lies in the value of $w(0)$ computed from Eq. (37). This is an exact value if $u(x)$ is exact. However, the solution is an approximate solution which will presumably deviate from the exact solution. We thus expect that $w(0)$ computed from Eq. (37), even with the exact $u(x)$, will not be the correct $w(0)$ for this problem. Note furthermore that a small error in $u(x)$ is multiplied fourfold in the computation of Eq. (37).

SECTION III

HIGHER-ORDER DIFFERENTIAL APPROXIMATIONS

Considerable effort has been directed towards obtaining solutions of the radiative transfer equation and its close counterpart in neutron transport theory. A thorough literature search carried out indicates that this effort can be subdivided into the following major categories:

a. Exact solutions

b. Purely numerical solutions, including the Monte Carlo technique

c. Approximation schemes

Because of its complicated integro-differential nature, exact solutions have been obtained only in very simple cases with restrictions on the absorption coefficient and geometry of the system. Together with the second category (numerical solution) the exact solutions are of interest in so far as they provide a check on the results from approximate techniques of solution.

Most of the approximation schemes are related to one or the other of two general methods for approximating the angular dependence of the radiation intensity — the spherical harmonics method (or the P_n method) and the discrete ordinates method (or the S_n method). Interest in the latter method has been very limited and we have primarily concerned ourselves with the P_n and related methods in our effort.

The straightforward spherical harmonics approximation consists of expanding the radiation intensity I in terms of spherical harmonics Y_n^m (which reduce to Legendre polynomials in planar and spherically symmetric geometries and to associated Legendre polynomials in a cylindrically symmetric case), substituting into the transfer equation for I, multiplying the resulting equation by $\bar{Y}_n^m$, the conjugate spherical harmonics, and integrating over the complete solid angle range. Use of the orthogonality and recurrence relations for the spherical harmonics leads to an infinite series of ordinary differential equations in terms of the coefficients of the spherical harmonics expansion, A_n^m, which are functions of position only and are thereby related to the quantities of interest, viz., the radiation intensity and heat flux vector.

The P_n approximation consists of setting the coefficients $A_n^m = 0$ for $n \geq N+1$, thus leading to a finite, determinate set of ordinary differential equations. The first approximation or P_1 approximation is often referred to as the differential approximation.

Although, as formulated, this theory has no explicit restrictions on either absorption coefficient or geometry, it turns out that, as a practical matter, the P_1 or P_3 approximations fail whenever the directional distribution of the radiation intensity becomes singular or when any volume element in the gas is not traversed by photons coming from all directions. Such is the case for the radiation field in curvilinear geometrics such as concentric spheres or cylinders.

Several techniques for overcoming this problem have been published in recent years. One of the more promising of these is that by Traugott[4] in which a modified spherical harmonic technique is applied to the radiation field between concentric spheres. Instead of developing the straight-forward spherical harmonics approximation, Traugott develops a finite set of moment equations where the moments are those of I with respect to direction cosines of $\vec{\Omega}$. As a system of moment equations is indeterminate, it is necessary to obtain closure relations between the moments with the aid of a spherical harmonics expansion of the intensity. It is in these closure relations that the modification is made by requiring that these relations hold not only for the case of isotropy but also for a unidirectional (radial) intensity field.

The attractiveness of this approach lies in the fact that it is possible to incorporate into the approximation scheme physical arguments relevant to the particular geometry under study. With this in view, our main effort this quarter has been towards developing general moment equations and closure relations between moments in cylindrical geometry. These results are summarized in the following sections.

1. Transfer Equation

The radiative transfer equation for a non-scattering, steady-state gas in local thermodynamic equilibrium can be written as[5]

$$\vec{\Omega} \cdot \vec{\nabla} I_\nu(\vec{r},\vec{\Omega}) = - \varkappa_\nu(T) \; (I_\nu(\vec{r},\vec{\Omega}) - B_\nu(T)) \tag{38}$$

where $\vec{\Omega}$ is a unit vector in the direction of photon propagation, I_ν is the spectral radiation intensity, B_ν is the Planck function for

equilibrium radiation, $\varkappa_\nu$ is the effective absorption coefficient and T is temperature. The Planck function B_ν is defined by

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{(\exp(h\nu/kT)-1)} \tag{39}$$

Furthermore, the radiative heat flux vector — the quantity of main interest to us — is defined by

$$\vec{q}_\nu = \int_{4\pi} I_\nu \vec{\Omega}\, d\,\vec{\Omega} \tag{40}$$

For the sake of convenience, we shall suppress the functional dependence of these quantities as well as the subscript "ν" .

It is convenient to rewrite the transfer equation in terms of cylindrical polar coordinates at the outset. Using the coordinate system shown in Fig. 2 and assuming cylindrical symmetry, we find for Eqn. (38):

$$\sin\theta \cos\chi \frac{\partial I}{\partial r} - \frac{\sin\theta \sin\chi}{r} \frac{\partial I}{\partial \chi} = -\varkappa(I-B) \tag{41a}$$

Alternatively, this can be written in terms of direction cosines as

$$\ell_r \frac{\partial I}{\partial r} - \frac{\ell_\varpi}{r} \frac{\partial I}{\partial \chi} = -\varkappa(I-B) \tag{41b}$$

where the direction cosines are given by

$$\ell_r = \sin\theta \cos\chi = (1-\mu^2)^{1/2} \cos\chi \tag{42a}$$

$$\ell_\varpi = \sin\theta \sin\chi = (1-\mu^2)^{1/2} \sin\chi \tag{42b}$$

$$\ell_z = \cos\theta = \mu \tag{42c}$$

2. Moment Equations

We define a sequence of moments of the intensity as follows:

$$I_{000} = \int_{4\pi} I d\Omega \tag{43a}$$

$$I_{100} = \int_{4\pi} I \ell_r d\Omega \tag{43b}$$

$$I_{010} = \int_{4\pi} I \ell_\varpi d\Omega \tag{43c}$$

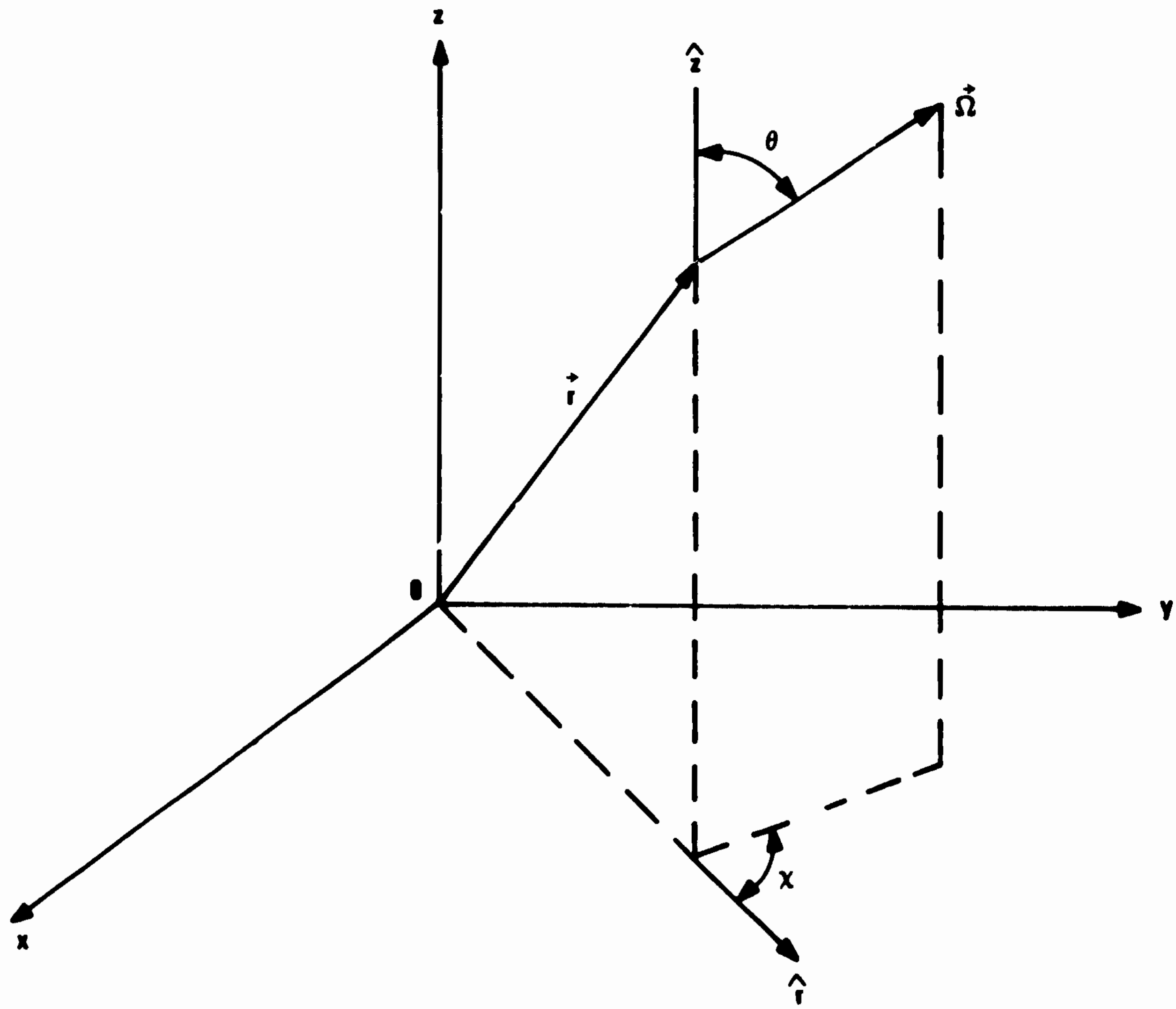

Figure 2. Coordinate System in Cylindrical Geometry (z corresponds to axis of cylinder).

$$I_{001} = \int_{4\pi} I \ell_z \, d\Omega \tag{43d}$$

$$I_{110} = \int_{4\pi} I \ell_r \ell_\varphi \, d\Omega \tag{43e}$$

$$I_{101} = \int_{4\pi} I \ell_r \ell_z \, d\Omega \tag{43f}$$

$$I_{011} = \int_{4\pi} I \ell_\varphi \ell_z \, d\Omega \tag{43g}$$

Here, it is understood that the first, second and third subscripts refer to the order of ℓ_r , ℓ_φ and ℓ_z , respectively, in the integrand on the R.H.S.

It is easily shown that the zeroth order moment I_{000} is proportional to the radiation energy density $u(r)$, the three first-order moments $(I_{100}, I_{010}, I_{001})$ comprise the three components of the radiation heat flux vector $\vec{q}$ and the nine second-order moments are proportional to the radiation pressure tensor. Moments higher than second-order have no physical significance.

Differential equations for the moments are derived as follows: To obtain the nth order moment equation we multiply the transfer equation (41b) by ℓ_i^n (where ℓ_i can be ℓ_r , ℓ_φ or ℓ_z) and integrate the resulting equation with respect to direction over the entire solid angle range. Thus the zeroth order moment equation is

$$\frac{dI_{100}}{dr} + \frac{1}{r} \; I_{100} + \kappa \, I_{000} = 4\pi\kappa B \tag{44}$$

The first-order moment equations consist of a set of three equations as follows:

$$\frac{dI_{200}}{dr} + \frac{1}{r} \left(I_{200} - I_{020}\right) + \kappa I_{100} = 0 \tag{45a}$$

$$\frac{dI_{110}}{dr} + \frac{2I_{110}}{r} + \kappa I_{010} = 0 \tag{45b}$$

$$\frac{dI_{101}}{dr} + \frac{I_{101}}{r} + \kappa I_{001} = 0 \tag{45c}$$

By continuing this process we can generate an infinite number of moment equations which, taken together, are completely equivalent to the original transfer equation. An approximation consists of truncating this series and

considering only a finite number of moments equations. In this case, however, the number of unknowns usually exceeds the number of equations, and it is necessary to obtain additional relations between the moments so that the system becomes determinate.

We can obtain these "closure" relations as follows. Expand the intensity in a series of associated Legendre polynomials:

$$I(r,\mu,\chi) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \left[P_\ell^0(\mu) A_\ell^0(r) + 2 \sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\mu) \cos\chi \, A_m^m(r) \right] \tag{46}$$

For the Nth order moment approximation scheme (which is obtained by integration w.r.t. $\vec{\Omega}$ of successive products of (41b) and ℓ_1^m, m = 0,1,2...N) we truncate the expansion (46) at $\ell = N$, by setting $A_\ell^m = 0$ for $\ell \geq N+1$. Substitution of the truncated spherical harmonics series for I into the defining equations for the moments (43) and elimination of the coefficients A_ℓ^m from the resulting equations give us the necessary number of closure relations between the moments to render the problem determinate. This resulting problem — the moment equations plus the closure relations — is identical to the determinate system of differential equations in terms of the coefficients A_m^ℓ one obtains by carrying out the straightforward spherical harmonics expansion (described in the introduction to the section) to order P_N.

For the first-order approximation scheme, in which equations (44) and (45) are the moment differential equations, the closure relations are found to be

$$I_{010} = I_{110} = I_{101} = 0 \tag{47a}$$

$$I_{200} = I_{020} = I_{002} = 1/3 \; I_{000} \tag{47b}$$

Substitution of (47) into (44), (45) leads to

$$\frac{dI_{100}}{dr} + \frac{1}{r} I_{100} + \kappa I_{000} = 4\pi \kappa B \tag{48a}$$

$$\frac{dI_{000}}{dr} + 3\kappa I_{100} = 0 \tag{48b}$$

Replacing I_{100} by q_r, the radial component of the radiation heat flux vector, and I_{000} by cu (where u is the radiation energy density), we reproduce equations (3)-(4) of the earlier section. In other words, the first-order moment approximation scheme is identical to the differential or P_1 approximation.

We have evaluated the differential equations and the closure relations for the next two higher moment approximation schemes. These correspond to the P_2 and P_3 spherical harmonics approximations. As noted in the Introduction, we are attempting to modify the closure relations in these higher approximations by incorporating physical arguments relevant to a curvilinear geometry. For the P_3 approximation, which is being developed at present, we are modifying the closure relations so that they satisfy the relationship between moments that result when the radiation field consists of an undirectional radial beam. In this way the resultant differential equations should be able to more accurately predict the radiative heat flux.

In these approximate schemes for solution of the transfer equation it is necessary to replace the exact boundary conditions by those that are consistent with the approximation being made. Two general techniques have been developed for this purpose and have been applied extensively in neutron transport problems.[4] One of our primary goals was to develop a consistent set of boundary conditions for the higher-order approximation schemes discussed, and then to apply the differential equations plus boundary conditions to the arc problem.

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